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Statistical Model Checking for Rare Events

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1. Problem

Verification of Stochastic Systems

- Verification of stochastic system models via statistical model checking

- Temporal logic specification:
"in the next 20 min. the system is unavailable for 1sec"

- If Φ = "in the next 20 min. the system is unavailable for 1 sec."

Probability (Φ) = ?

- Equivalently: A biased coin (Bernoulli random variable)
- Prob (Heads) = *p*   Prob (Tails) = *1-p*
  - *p* is unknown
- Question: What is *p*?
- A solution: flip the coin a number of times, collect the outcomes, and use statistical estimation

2. Statistical Model Checking

Key idea

(Haakan Younes, 2001)

- System behavior w.r.t. property Φ can be modeled by a Bernoulli random variable of parameter *p*:
  - System satisfies Φ with (unknown) probability *p*
- Question: What is *p?*
- Draw a sample of system simulations and use:
  - Statistical estimation: returns "*p* in interval (a,b)" with high probability

3. Temporal Logic

- A formal notation for expressing properties about the temporal evolution of a system
- Example: "*within 10 time units the system will shut down and the shutdown signal will be ON until then*"

`shutdown_ON` $\mathbf{U}^{10}$ `sysdown`

- Example: "*it is not the case that in the future 25 time units the system is globally down for one time unit*"

$\neg(\mathbf{F}^{25}\,\mathbf{G}^{1}$ `sysdown`$)$

4. Bounded Linear Temporal Logic

- Extension of LTL with time bounds on temporal operators.
  - No neXt operator
- Let $\sigma = (s_0, t_0), (s_1, t_1), \ldots$ be a trace of the model
  - the system stays in state s_i for time t_i
- The semantics of BLTL for trace σ starting at state k (σ^k):
  - $\sigma^k \models ap$ iff atomic proposition ap true in state s_k
  - $\sigma^k \models \Phi_1 \vee \Phi_2$ iff $\sigma^k \models \Phi_1$ or $\sigma^k \models \Phi_2$
  - $\sigma^k \models \neg\Phi$ iff $\sigma^k \models \Phi$ does not hold
  - $\sigma^k \models \Phi_1\, \mathbf{U}^t\, \Phi_2$ iff there exists natural i such that
    1) $\sigma^{k+i} \models \Phi_2$
    2) $\Sigma_{j<i}\, t_{k+j} \leq t$
    3) for each $0 \leq j < i$, $\sigma^{k+j} \models \Phi_1$

"within time t, Φ_2 will be true and Φ_1 will hold until then"

- In particular, $\mathbf{F}^t\,\Phi = true\ \mathbf{U}^t\,\Phi$, $\mathbf{G}^t\,\Phi = \neg\mathbf{F}^t\,\neg\Phi$

- *Definition:* The time bound of Φ:
  - $\#(ap) = 0$
  - $\#(\neg\Phi) = \#(\Phi)$
  - $\#(\Phi_1 \vee \Phi_2) = \max(\#(\Phi_1), \#(\Phi_2))$
  - $\#(\Phi_1\, \mathbf{U}^t\, \Phi_2) = t + \max(\#(\Phi_1), \#(\Phi_2))$
- *Lemma:* "Bounded simulations suffice"

Let Φ be a BLTL *property, and k≥0. For any two infinite traces ρ, σ such that* ρ^k *and* σ^k *"equal up to time* $\#(\Phi)$*" we have*

$$\rho^k \models \Phi \quad iff \quad \sigma^k \models \Phi$$

5. Rare Events

- Estimate Prob(X≥*t*) = *p*, when *p* is *small* (say 10^{-9})
- Standard (Crude) Monte Carlo: generate *K* i.i.d. samples of X; return the estimator e_K

$$e_K = \frac{1}{K}\sum_{i=1}^{K} I(X_i \geqslant t) = \frac{k_t}{K}$$

- Prob ($e_K \to p$) = 1 for $K \to \infty$ (strong law LN)
- Relative Error (RE) = $\dfrac{\sqrt{\mathrm{var}[e_K]}}{\mathrm{E}[e_K]} = \dfrac{\sqrt{p(1-p)}}{p\sqrt{K}}$
- More accuracy → more samples
- Want confidence interval of relative accuracy δ and coverage probability *c,* i.e., estimate e_K must satisfy:

$$\mathrm{Prob}(|\, e_K - p\,| < \delta \cdot p) \geq c$$

- From the CLT, a 99% (approximate) confidence interval of relative accuracy δ needs about

$$K \approx \frac{1-p}{p\delta^2} \text{ samples } \quad (\to \mathrm{Prob}(|\, e_K - p\,| < \delta p) \approx 0.99)$$

- Example: $p = 10^{-9}$ and $\delta = 10^{-2}$ (*ie*, 1% relative accuracy) we need about $\mathbf{10^{13}}$ **samples**!!

6. Importance Sampling

$$\begin{aligned} p &= E[I(X \geqslant t)] \\ &= \int I(x \geqslant t) f(x)\, dx \\ &= \int I(x \geqslant t) \boxed{\frac{f(x)}{f_*(x)}} f_*(x)\, dx \\ &= \int I(x \geqslant t) W(x) f_*(x)\, dx \\ &= E_*[I(X \geqslant t) W(X)] \end{aligned}$$

where *f* is the density of *X*.

- The Importance Sampling estimator is:

$$p_K = \frac{1}{K}\sum_{i=1}^{K} I(X_i \geqslant t) W(X_i), \qquad \boxed{X_i \sim f_*}$$

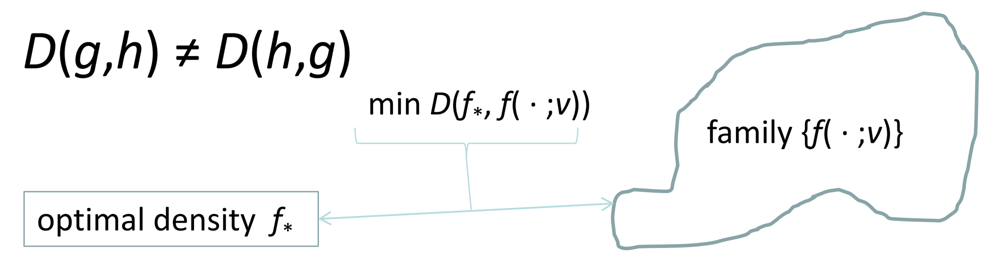
- Need to choose a "good" biasing density (low variance)
- Optimal density: $f_*(x) = \dfrac{I(x \geq t)\, f(x)}{p}$
- Idea: search for a "good density" in a parameterized family

7. Cross-Entropy

- The cross-entropy of densities *g*, *h* is

$$D(g,h) = E_g\left[\ln \frac{g(X)}{h(X)}\right] = \int g(x) \ln g(x) dx - \int g(x) \ln h(x) dx$$

  - $D(g,h) \geq 0$ (= 0 IFF $g = h$)
  - $D(g,h) \neq D(h,g)$



- The Cross-Entropy Method has two steps

1. find $v_* = \underset{v}{\arg\min}\, D(f_*(\cdot),\, f(\cdot\,; v))$
2. run importance sampling with biasing density $f(\cdot\,; v_*)$

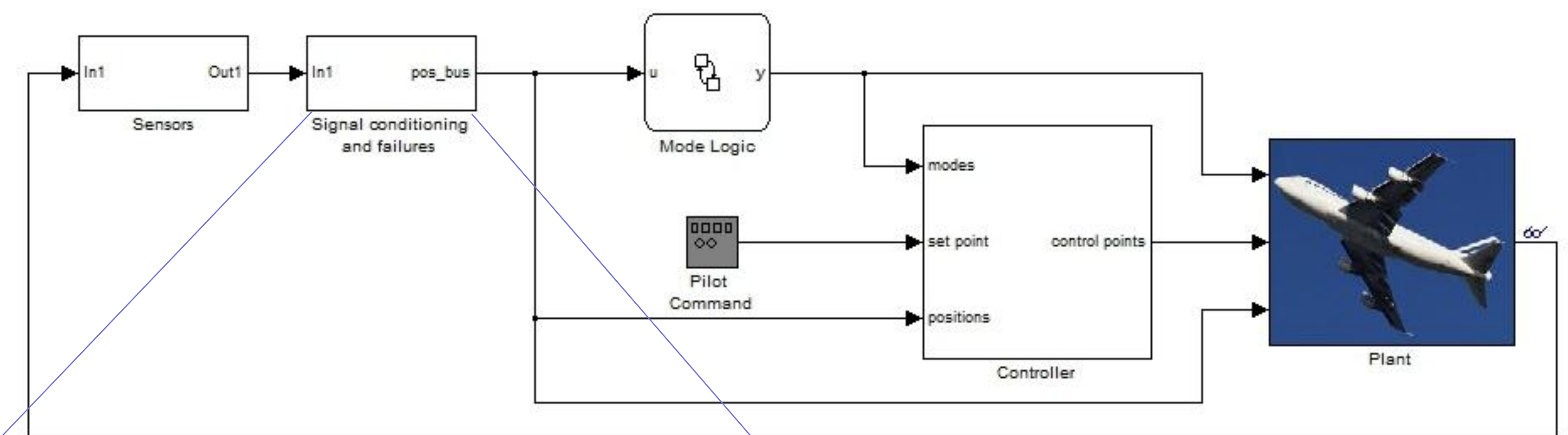
- We can estimate v_* by Monte Carlo simulation

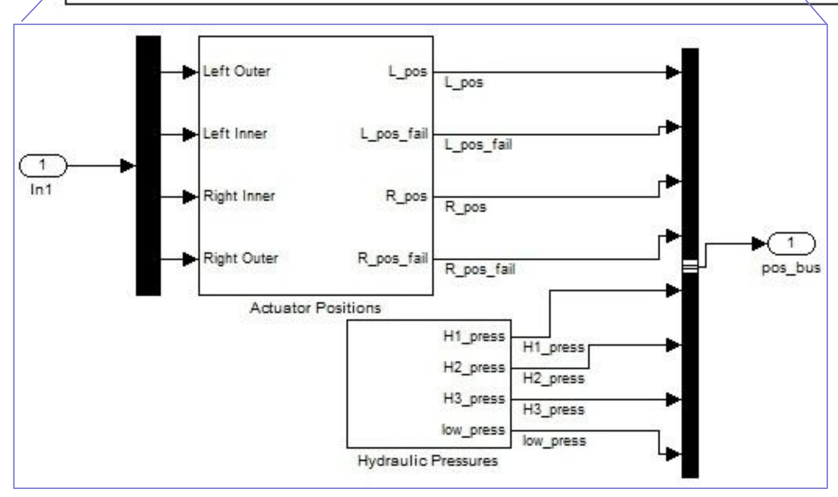
$$\bar{v}_* = \frac{\sum_{i=1}^{K} [I(X_i \geq t) W(X_i; u, w) X_i]}{\sum_{i=1}^{K} [I(X_i \geq t) W(X_i; u, w)]}$$

where $X_1, \ldots, X_K$ samples iid as $f(\cdot\,; w)$

8. Applications

- Fault-tolerant controller for an aircraft elevator system





- The three hydraulic circuits can independently fail
- What is the probability that in the next 25s for 1s no control input is passed to the elevators?

Prob($\mathbf{F}^{25}\,\mathbf{G}^{1}$ (H1_*fail* or H3_*fail*) and H2_*fail*)=?

	Samples	Estimate	Relative error	Time (h)
Samples	Step 1: 100 Step 2: 1,000	1.58 x 10^{-14}	0.58	0.23
	Step 1: 1,000 Step 2: 10,000	8.54 x 10^{-14}	0.24	2.45
	Step 1: 10,000 Step 2: 100,000	8.11 x 10^{-14}	0.17	23.9

9. References

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- P. Zuliani, C. Baier, E.M. Clarke. *Rare-Event Verification for Stochastic Hybrid Systems*. Submitted